ABSTRACT

In this work, using the Calkin-Gorbachuk method firstly all selfadjoint extensions of the minimal operator generated by first order linear singular quasi-differential expression in the weighted Hilbert space of vector-functions on right semi-axis have been described. Lastly, the structure of the spectrum set of these extensions has been investigated.

Keywords:
Selfadjoint operator; Quasi-differential operator; Spectrum.

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47A10, 47B25

INTRODUCTION

In the first years of previous century, J. von Neumann [13] and M. H. Stone [12] investigated the theory of selfadjoint extensions of linear densely defined closed symmetric operators in a Hilbert spaces. Applications to scalar linear even order symmetric differential operators and description of all selfadjoint extensions in terms of boundary conditions were done by I. M. Glazman in his seminal work [5] and by M. A. Naimark [10] in his book. In this sense the famous Glazman-Krein-Naimark (or Everitt-Krein-Glazman-Naimark) Theorem in mathematical literature should be noted. In mathematical literature there is another so-called Calkin-Gorbachuk method (see [6], [11]).

Our motivation in this paper originates from the interesting researches of W. N. Everitt, L. Markus, A. Zettl, J. Sun, D. O’Regan, R. Agarwal [2], [3], [4], [14] in scalar cases. Throughout this paper A. Zettl’s and J. Sun’s view about these topics is to be taken into consideration in [14]: A selfadjoint ordinary differential operator in Hilbert space is generated by two things:

1. a symmetric (formally selfadjoint) differential expression;

2. a boundary condition which determined selfadjoint differential operators.

And also for a given selfadjoint differential operator, a basic question is: What is its spectrum?

In this work, in Section 3 the representation of all selfadjoint extensions of a symmetric quasi-differential operator, generated by first order symmetric quasi-differential expression in the weighted Hilbert space of vector-functions defined at the right semi-axis in terms of boundary conditions have been described. In Section 4, the structure of spectrum of these selfadjoint extensions is investigated.

STATEMENT OF THE PROBLEM

In the weighted Hilbert space \( L^2_+(H,(a,\infty)) \) where \( H \) is a separable Hilbert space and \( a \in \mathbb{R} \), we will consider the following quasi-differential expression given by

\[
I(u) = \frac{\alpha(t)}{w(t)}(au(t))' + Au(t).
\]

Here

1. \( \alpha, w : (a, \infty) \to (0, \infty) \),
2. \( \alpha, w \in C(a, \infty) \),
3. \( \int_a^\infty \frac{w(s)}{\alpha^2(s)} \, ds < \infty \),
4. \( A : D(A) \subset H \to H \) is a selfadjoint operator.
In this case, since
\[ (I(u),v)_{L^2_0(H,(a,\infty))} \]
\[ = \int_a^\infty \left( \frac{\alpha(t)}{w(t)} (au)(t),v(t) \right)_H w(t) \, dt + (Au,v)_{L^2_0(H,(a,\infty))} \]
then the differential-operator expression \( I(\cdot) \) is formally symmetric.

The minimal \( L_0 \) and maximal \( L \) operators corresponding to differential-operator expression in \( L^2_0(H,(a,\infty)) \) can be defined by using the classical techniques (see [7]).

On the other hand one can easily see that
\[
D(L) = \{ u \in L^2_0(H,(a,\infty)) : (I(u)) \in L^2_0(H,(a,\infty)) \},
\]
\[
D(L_0) = \{ u \in D(L) : (au)(a) = (au)(\infty) = 0 \}.
\]

DESCRIPTION OF SELFADJOINT EXTENSIONS

In this section using the Calkin-Gorbachuk method we will investigate the general representation of all selfadjoint extensions of the minimal operator \( L_0 \).

First, let us prove the following assertion.

**Lemma 3.1** The deficiency indices of the minimal operator \( L_0 \) in \( L^2_0(H,(a,\infty)) \) are in form
\[
(n_0(L_0),n_0(L_0)) = (\dim H, \dim H).
\]

**Proof.** For the simplicity of calculations it will be taken \( A=0 \). It is clear that the general solutions of differential equations
\[ \frac{\alpha(t)}{w(t)}(au)(t) \pm \alpha(t) = 0, t > a \]
be can given as
\[ u_\pm(t) = \frac{1}{\alpha(t)} \exp \left( \int_a^t \frac{w(s)}{\alpha(s)} \, ds \right) f, f \in H, t > a. \]

From these representations we have
\[
\| u \|_{L^2_0(H,(a,\infty))}^2 \]
\[ = \int_a^\infty \left( \right) \, dt \]
\[ = \int_a^\infty \left( \right) \, \| f \|_{L^2_0}^2 \]
\[ = \int_a^\infty \left( \right) \, \| f \|_{L^2_0}^2 \]
\[ = \frac{1}{2} \left( \right) \, \| f \|_{L^2_0}^2 < \infty. \]

Consequently, \( n_0(L_0) = \dim \text{ker}(L-iE) = \dim H. \)

On the other hand, it is clear that for any \( f \in H \), one can obtain
\[
\| u \|_{L^2_0(H,(a,\infty))}^2 \]
\[ = \int_a^\infty \left( \right) \, dt \]
\[ = \int_a^\infty \left( \right) \, \| f \|_{L^2_0}^2 \]
\[ = \int_a^\infty \left( \right) \, \| f \|_{L^2_0}^2 \]
\[ = \frac{1}{2} \left( \right) \, \| f \|_{L^2_0}^2 < \infty. \]

Consequently, \( n(L_0) = \dim \text{ker}(L+iE) = \dim H. \) This completes the proof.

As a result, the minimal operator \( L_0 \) has at least one selfadjoint extension (see [6]).

In order to describe these extensions we need to obtain the space of boundary values.

**Definition 3.2** [6] Let \( H \) be any Hilbert space and \( S : D(S) \subset H \rightarrow H \) be a closed densely defined symmetric operator in the Hilbert space \( H \) having equal finite or infinite deficiency indices. A triplet \( (H,\gamma_1,\gamma_2) \) where \( H \) is a Hilbert space, \( \gamma_1 \) and \( \gamma_2 \) are linear mappings from \( D(S^*) \) into \( H, \) is called a space of boundary values for the operator \( S \) if for any \( f,g \in D(S^*) \)
\[ \langle S^*f,g \rangle_N = \langle f,S^*g \rangle_N \]
\[ = \langle \gamma_1(f),\gamma_2(g) \rangle_H - \langle \gamma_2(f),\gamma_1(g) \rangle_H \]
while for any \( F_1,F_2 \in H, \) there exists an element \( f \in D(S^*) \)
such that $\gamma_1(f) = F_1$ and $\gamma_2(f) = F_2$.

**Lemma 3.3** The triplet $(H, \gamma_1, \gamma_2)$,

\[
\gamma_1 : D(L) \to H, \gamma_1(u) = \frac{1}{\sqrt{2}}(\langle au \rangle(\infty) - \langle au \rangle(a)) ,
\]

\[
\gamma_2 : D(L) \to H, \gamma_2(u) = \frac{1}{\sqrt{2}}(\langle au \rangle(\infty) + \langle au \rangle(a))
\]

is a space of boundary values of the minimal operator $L_\alpha$ in $L^2(H,(a,\infty))$. 

**Proof.** For any $u,v \in D(L)$,

\[
( Lu , v )_{L^2(H,(a,\infty))} - ( u , Lv )_{L^2(H,(a,\infty))} = \left( \alpha(t) \langle au \rangle, \alpha(t) v(t) \right)_H t dt
\]

then it is clear that $u \in D(L)$ and $\gamma_1(u) = \gamma_2(u) = g$.

The following result can be established by using the method given in [6].

**Theorem 3.4** If $L$ is a selfadjoint extension of the minimal operator $L_\alpha$ in $L^2(H,(a,\infty))$, then it is generated by the differential-operator expression $l(\cdot)$ and boundary condition

\[
\langle au \rangle(\infty) = W(\langle au \rangle(\infty)),
\]

where $W : H \to H$ is an unitary operator. Moreover, the unitary operator $W$ in $H$ is determined uniquely by the extension $L$, i.e. $L = L_\alpha$ and vice versa.

**Proof.** It is known that all selfadjoint extension of the minimal operator $L_\alpha$ are described by the differential-operator expression $l(\cdot)$ with boundary condition

\[
(V - E)\gamma_1(u) + i(V + E)\gamma_2(u) = 0,
\]

where $V : H \to H$ is an unitary operator. Therefore from Lemma 3.3 we obtain

\[
(V - E)(\langle au \rangle(\infty) - \langle au \rangle(a)) + (V + E)(\langle au \rangle(\infty) + \langle au \rangle(a)) = 0, u \in D(L).
\]

From this, it is implies that

\[
\langle au \rangle(a) = -V(\langle au \rangle(\infty)).
\]

Choosing $W = V$ in last boundary condition, we have

\[
\langle au \rangle(a) = W(\langle au \rangle(\infty)).
\]

**THE SPECTRUM OF THE SELFADJOINT EXTENSIONS**

In this section the structure of the spectrum set of the selfadjoint extensions of the minimal operator $L_\alpha$ in $L^2(H,(a,\infty))$ will be examined.

**Theorem 4.1** The spectrum of any selfadjoint extension $L_\mu$ is in form

\[
\sigma(L_\mu) = \lambda \in \mathbb{R} : \lambda = \left( \int_{\mathbb{R}} \frac{w}{\alpha^2(s)} ds \right)^{-1} (arg \mu + 2n\pi)
\]

\[
\mu \in \mathbb{W} \exp \left( i \int_{\mathbb{R}} \frac{w(s)}{\alpha(s)} ds \right), n \in \mathbb{Z}.
\]

**Proof.** Consider the following problem to spectrum of the extension $L_\mu$, i.e.

\[
l(u) = \lambda u + f, u \in L^2(H,(a,\infty)), \lambda \in \mathbb{R},
\]

\[
\langle au \rangle(a) = W(\langle au \rangle(\infty)).
\]
that is,
\[
\frac{\alpha(t)}{w(t)} (\alpha u)(t) + Au(t) = \lambda u(t) + f(t), t > a,
\]
\[
(\alpha u)(a) = W(\alpha u)(\infty).
\]

The general solution of the last differential equation,
\[
(\alpha u)'(t) = i \frac{w(t)}{\alpha(t)}(A - \lambda E)(\alpha u)(t) - i \frac{w(t)}{\alpha(t)} f(t)
\]
is in form
\[
u(t, \lambda) = \frac{1}{\alpha(t)} \exp \left( i \int_{\infty}^{t} \frac{w(s)}{\alpha^2(s)} ds \right) f_i + i \frac{w(t)}{\alpha(t)} \int_{\infty}^{t} \frac{w(\tau)}{\alpha^2(\tau)} f(s) ds,
\]
f_i \in H, t > a.

In this case
\[
\left\| \frac{1}{\alpha(t)} \exp \left( i \int_{\infty}^{t} \frac{w(s)}{\alpha^2(s)} ds \right) f_i \right\|_{L^2(H, (a, \infty))} < \infty
\]
and
\[
\left\| \frac{1}{\alpha(t)} \exp \left( i \int_{\infty}^{t} \frac{w(\tau)}{\alpha^2(\tau)} d\tau \right) \frac{w(s)}{\alpha(s)} f(s) ds \right\|_{L^2(H, (a, \infty))} < \infty
\]
\[
= -i \exp \left( i \lambda_0 \int_{\infty}^{t} \frac{w(\tau)}{\alpha^2(\tau)} f(s) ds \right).
\]

Therefore in order to \( \lambda \in \sigma(L_w) \) the necessary and sufficient condition is
\[
\exp \left( i \lambda_0 \int_{\infty}^{t} \frac{w(s)}{\alpha^2(s)} ds \right) = \mu \in \sigma \left( W \exp \left( i A \int_{\infty}^{t} \frac{w(s)}{\alpha^2(s)} ds \right) \right).
\]

Since the operator
\[
W \exp \left( i A \int_{\infty}^{t} \frac{w(s)}{\alpha^2(s)} ds \right)
\]
is isometric, then \( |\mu| = 1 \).

Consequently,
\[
\lambda_0 \int_{\infty}^{t} \frac{w(s)}{\alpha^2(s)} ds = \arg \mu + 2n\pi, n \in \mathbb{Z}.
\]

On the other hand since \( \int_{\infty}^{t} \frac{w(s)}{\alpha^2(s)} ds > 0 \), then
\[
\lambda = \left( \frac{\int_{\infty}^{t} w(s)}{\alpha^2(s)} ds \right)^{-1} (\arg \mu + 2n\pi),
\]
\[\mu \in \sigma \left( W \exp \left( i A \int_{\infty}^{t} \frac{w(s)}{\alpha^2(s)} ds \right) \right), n \in \mathbb{Z}.
\]

This completes the proof.

**Remark 4.2** Note that the similar problems in different singular multipoint cases in the corresponding direct sums of Hilbert spaces of vector-functions have been investigated in [1], [8], [9].

**References**

5. Glazman IM. On the theory of singular differential operators.


